PMM U.S.S.R.,Vol.46,pp.536-542 0021-8928/83/4 0536 $7.50/0

UDC 539.376

CONTACT PROBLEM FOR A TWO-LAYER AGING VISCOELASTIC FOUNDATION*

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A solution is given for the problem of the frictionless impression of a stamp into a two-layered aging viscoelastic strip in the case of plane strain. The upper layer is thin. The lower layer is hinge-fixed along the foundation. It is assumed that the layers are in contact without friction, the force acting on the stamp and the domain of contact do not change with lapse of time, the rheological properties of the layers are described by the equations of the linear creep theory of aging materials, and the layers are fabricated at different times.

By using the Fourier integral transform in the longitudinal coordinate and the principle of correspondence in the linear creep theory of aging media, the problem under consideration is reduced to the determination of unknown contact stresses under the stamp from an integral equation of the second kind containing Fredholm and Volterra operators. In the general case, the solution of the equation obtained is constructed by asymptotic method for relatively large times. In particular, when the hereditary properties of the layer materials are identical, expansions are found for the fundamental characteristics of the phenomenon, which are valid in the whole range of time variation.

1. Let a layer $0 \leqslant y \leqslant h$ be hinge adherent to the surface of a layer of large thickness H lying without friction on an undeformable foundation. We assume a rigid stamp is impressed without friction by a force P with eccentricity of application e on the upper boundary of such a composite medium.

We associate the coordinate system xOy with the two-layer packet, and $x'O'y'$ with the stamp. The surface of the stamp foundation is given in the $x'y'$ axes by the function $y' = g(x')$, and the line of contact is determined by the inequality $|x'| \leqslant a$ (Fig.1).

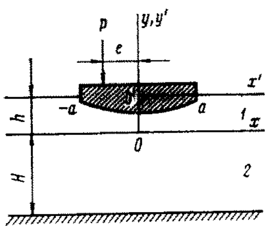

Fig.1

The rheological properties of the two-layer foundation will be described by the linear creep theory equations of aging materials /1/ (the numbers $k = 1, 2$ are appended to each layer counting from the top down)

$$e_{ij}^{(k)}(t) = (1 + \nu^{(k)})\left[\frac{s_{ij}^{(k)}(t)}{E^{(k)}} - \int_{\tau_0}^{t} s_{ij}^{(k)}(\tau) K^{(k)}(t - \tau_1^{(k)}, \tau - \tau_1^{(k)})\, d\tau\right] \tag{1.1}$$

$$\varepsilon(t) = (1 - 2\nu^{(k)})\left[\frac{\sigma^{(k)}(t)}{E^{(k)}} - \int_{\tau_0}^{t} \sigma^{(k)}(\tau) K^{(k)}(t - \tau_1^{(k)}, \tau - \tau_1^{(k)})\, d\tau\right]$$

$$K^{(k)}(t, \tau) = \frac{\partial}{\partial \tau} C^{(k)}(t, \tau)$$

Here $s_{ij}^{(k)}(t)$, $e_{ij}^{(k)}(t)$ are the stress and strain tensor deviators, $3\varepsilon^{(k)}(t)$ is the volume strain, $\sigma^{(k)}(t)$ is the mean hydrostatic pressure, $K^{(k)}(t, \tau)$ is the creep kernel for the uniaxial state of stress, $C^{(k)}(t, \tau)$ is the measure of the creep, τ_0 is the time of application of the stress to the element of the aging viscoelastic medium, $\tau_1^{(k)}$ is the time of fabrication of this element. It is taken into account in (1.1) that $\nu^{(k)}$, the Poisson's ratio, and $E^{(k)}$, the modulus of instantaneous elastic strain of the material of the k-th layer, are independent of time.

We shall examine the properties of the function $C^{(k)}(t, \tau)$. It is known (**) that the measure of creep $C^{(k)}(t, \tau)$ can, under natural aging contitions when the material aging process is considered independent of the strain process, be represented as the product of two functions

*Prikl.Matem.Mekhan.,46,No.4,674-682,1982

**) Arutiunian, N.Kh. Theory of creep of inhomogeneously-aging bodies. Preprint of the Institute of Problems of Mechanics, Academy of Sciences of the USSR, Moscow, No.170, 1980.

$$C^{(k)}(t,\tau) = C^{(k)}(t-\tau,\tau) = \varphi^{(k)}(\tau) f^{(k)}(t-\tau) \tag{1.2}$$
$$\varphi^{(k)}(\tau) = \lim_{t\to\infty} C^{(k)}(t,\tau), \quad \forall\tau, \; C^{(k)}(t,t) \equiv 0$$

one of which $\varphi^{(k)}(\tau)$ takes account of the material aging process while the other $f^{(k)}(t-\tau)$ is the influence of the duration of its loading. The aging function $\varphi^{(k)}(\tau)$ is continuous, bounded, and decreases monotonically with the increase in age, tending to a certain constant $C_0^{(k)}$, which is the limit value of the measure of material creep in its old age. The function $f^{(k)}(t-\tau)$, characterizing the hereditary properties of the material, would vary within the limits $0 \leqslant f^{(k)}(t-\tau) \leqslant 1$ in the range $0 \leqslant t-\tau < \infty$. Approximating $f^{(k)}(t-\tau)$ by a finite set of exponential functions

$$f^{(k)}(t-\tau) = \sum_{j=0}^{N} B_j^{(k)} \exp[-\gamma_j^{(k)}(t-\tau)]$$
$$B_0^{(k)} = 1, \quad \sum_{j=0}^{N} B_j^{(k)} = 0, \quad \gamma_0^{(k)} = 0, \quad \gamma_j^{(k)} > 0 \quad (j = 1, 2, \ldots, N)$$

where $B_j^{(k)}$, $\gamma_j^{(k)}$ are constant parameters selected in a suitable manner for this material, we write in conformity with (1.1) and (1.2)

$$\lim_{t\to\infty} K^{(k)}(t,\tau) = [\varphi^{(k)}(\tau)]' \tag{1.3}$$

Now using the results of /2/ and the principle of correspondence /1/, we obtain an integral equation in the unknown contact pressures $q(x,t)$ under the stamp. Taking account of the smallness of the quantity ha^{-1} and the notation

$$\xi^* = \xi a^{-1}, \quad x^* = xa^{-1}, \quad t^* = t\tau_0^{-1}, \quad \theta^{(k)} = 0.5E^{(k)}[1-(\nu^{(k)})^2]^{-1}$$
$$c = 0.5ha^{-1}\pi\theta^{(2)}(\theta^{(1)})^{-1}, \quad (\tau_1^{(k)})^* = \tau_1^{(k)}\tau_0^{-1}, \quad q^*(x^*,t^*) =$$
$$(\theta^{(2)})^{-1} q(x,t), \quad E^{(k)}C^{(k)}(t,\tau) = [C^{(k)}(t^*,\tau^*)]^*, \quad g(x) = ag^*(x^*)$$

(we later omit the asterisk), we write it in the form

$$c\left[q(x,t) - \int_1^t q(x,\tau) K^{(1)}(t-\tau_1^{(1)}, \tau-\tau_1^{(1)})\, d\tau\right] + \left\{\int_{-1}^{1} q(\xi,t)[-\ln|\xi-x| + D]\, d\xi - \right. \tag{1.4}$$
$$\left.\int_1^t K^{(2)}(t-\tau_1^{(2)}, \tau-\tau_1^{(2)})\, d\tau \int_{-1}^{1} q(\xi,\tau)[-\ln|\xi-x| + D]\, d\xi\right\} =$$
$$\pi[\delta(t) + \alpha(t)x - g(x)]$$
$$|x| \leqslant 1, \; 1 \leqslant t \leqslant T < \infty, \; D = \ln(Ha^{-1}) - 0.352$$

Here $\delta(t) + \alpha(t)x$ is the rigid displacement of stamp under the action of the applied force P and the moment $M = Pe$.

The statics conditions

$$R_1 = P(a\theta^{(2)})^{-1} = \int_{-1}^{1} q(x,t)\, dx, \quad R_2 = Pe(a^2\theta^{(2)})^{-1} = \int_{-1}^{1} xq(x,t)\, dx \tag{1.5}$$

must be added to (1.4).

We note that for $t = 1$ equation (1.4) and conditions (1.5) acquire a form known from the theory of classical contact problems and correspond to the problem of impression of a stamp into an elastic strip of large thickness covered by Winkler springs /3/

$$cq(x,1) + \int_{-1}^{1} q(\xi,1)[-\ln|\xi-x| + D]\, d\xi = \pi[\delta(1) + \alpha(1)x - g(x)] \quad (|x| \leqslant 1) \tag{1.6}$$

$$R_1 = \int_{-1}^{1} q(x,1)\, dx, \quad R_2 = \int_{-1}^{1} xq(x,1)\, dx \tag{1.7}$$

Moreover, it is shown /3/ that if the function $g(x) \in L_2(-1, 1)$, then the solution of the integral equation (1.6) in the space $L_2(-1,1)$ exists and is unique for any value of the parameter $c \in (0, \infty)$.

2. Let us construct the solution of (1.4) for the problem formulated in the case $P =$ const. Without limiting the generality of the discussion, we study just the even variant ($g(x)$ is an even function of x, and $\alpha(t) \equiv 0$) by keeping in mind that everything can be done analogously for the odd case.

In conformity with the algorithm elucidated in /4/, let us examine the equivalent integral equation in place of (1.4):

$$c\left[q(x,t)-q(x,1)-\int_1^t q(x,\tau)K^{(1)}(t-\tau_1^{(1)},\tau-\tau_1^{(1)})\,d\tau\right]+ \tag{2.1}$$
$$\left\{\int_{-1}^{1}[q(\xi,t)-q(\xi,1)][-\ln|\xi-x|+D]\,d\xi-\right.$$
$$\left.\int_1^t K^{(2)}(t-\tau_1^{(2)},\tau-\tau_1^{(2)})\,d\tau\int_{-1}^{1}q(\xi,\tau)[-\ln|\xi-x|+D]\,d\xi\right\}=$$
$$\pi[\delta(t)-\delta(1)]\quad(|x|\leqslant 1,\quad 1\leqslant t\leqslant T<\infty)$$

and let us seek its solution in the form

$$q(x,t)=q_0(x)+\sum_{i=1}^{\infty}z_i(t)\,q_i(x) \tag{2.2}$$

Representing the function $\delta(t)$ that characterizes the rigid displacement of the stamp in the form /4/

$$\delta(t)=\delta y(t)+\delta_0+\sum_{i=1}^{\infty}\delta_i y_i(t) \tag{2.3}$$

where $\delta,\ \delta_j\ (j=0,1,\ldots,\ \ldots)$ are constants, we obtain

$$cq_0(x)\int_1^t K^{(1)}(t-\tau_1^{(1)},\tau-\tau_1^{(1)})\,d\tau+ \tag{2.4}$$
$$\int_1^t K^{(2)}(t-\tau_1^{(2)},\tau-\tau_1^{(2)})\,d\tau\int_{-1}^{1}q_0(\xi)[-\ln|\xi-x|+D]\,d\xi=$$
$$\pi\delta[y(1)-y(t)]$$

$$\int_1^t z_i(\tau)[K^{(2)}(t-\tau_1^{(2)},\tau-\tau_1^{(2)})+\alpha_i cK^{(1)}(t-\tau_1^{(1)},\tau-\tau_1^{(1)})]\,d\tau=(1+c\alpha_i)[z_i(t)-z_i(1)] \tag{2.5}$$

$$y_i(t)-y_i(1)=z_i(t)-z_i(1)-\int_1^t z_i(\tau)K^{(2)}(t-\tau_1^{(2)},\tau-\tau_1^{(2)})\,d\tau \tag{2.6}$$

$$\alpha_i Aq_i=q_i+\pi\alpha_i\delta_i,\quad A\varphi=\int_{-1}^{1}\varphi(\xi)[-\ln|\xi-x|+D]\,d\xi \tag{2.7}$$
$$|x|\leqslant 1,\quad 1\leqslant t\leqslant T<\infty,\quad i\geqslant 1$$

We note that for such a choice of the solution of the problem (see (2.2) and (2.3)), equation (2.4) is not successfully satisfied exactly in the general case. Taking into account the behavior of the creep kernel for large times (1.3), we satisfy (2.4) in the asymptotic sense as $t\to\infty$. We set

$$\int_1^t K^{(2)}(t-\tau_1^{(2)},\tau-\tau_1^{(2)})\,d\tau\approx F(\tau_1^{(1)},\tau_1^{(2)})\int_1^t K^{(1)}(t-\tau_1^{(1)},\tau-\tau_1^{(1)})\,d\tau$$
$$F(\tau_1^{(1)},\tau_1^{(2)})=\varphi^{(2)}(1-\tau_1^{(2)})[\varphi^{(1)}(1-\tau_1^{(1)})]^{-1}$$

(this relationship is satisfied exactly if the hereditary properties of the material of the layers are identical). Then

$$y(t)=-\int_1^t K^{(1)}(t-\tau_1^{(1)},\tau-\tau_1^{(1)})\,d\tau,\quad y(1)=0 \tag{2.8}$$

and $q_0(x)$ is determined from an integral equation of the type (1.6)

$$cq_0(x)+F(\tau_1^{(1)},\tau_1^{(2)})\int_{-1}^{1}q_0(\xi)[-\ln|\xi-x|+D]\,d\xi=\pi\delta\quad(|x|\leqslant 1) \tag{2.9}$$

for which the method of solution is elucidated in detail in /3/.

We examine the Fredholm integral equation (2.7) and we seek its solution in the form of a Fourier series in an orthonormalized system of Legendre polynomials

$$q_i(x) = \pi\sqrt{2}\,\alpha_i\delta_i \sum_0^\infty a_j^{(i)} P_{2j}^*(x), \quad P_j^*(x) = \sqrt{\frac{1+2j}{2}}\,P_j(x) \tag{2.10}$$

It is known /5/ that they form a basis in $L_2(-1, 1)$. Furthermore, expanding the kernel (2.7) in a double series in the mentioned system of polynomials

$$-\ln|\xi - x| + D = \sum_{m=0}^\infty \sum_{n=0}^\infty r_{mn} P_{2m}^*(\xi) P_{2n}^*(x) \tag{2.11}$$

substituting (2.10) and (2.11) into (2.7), using the property of orthogonality of the Legendre polynomials, and equating coefficients of the left and right sides for polynomials of identical number in the relationship obtained, we obtain (δ_{0n} is the Kronecker delta)

$$\alpha_i \sum_{j=0}^\infty a_j^{(i)} r_{nj} = a_n^{(i)} + \delta_{0n} \quad (i \geqslant 1, n = 0, 1, \ldots) \tag{2.12}$$

According to the inequality

$$\sum_{m=0}^\infty \sum_{n=0}^\infty r_{mn}^2 = B_1 < \infty, \quad B_1 = \text{const}$$

resulting from (2.11), it can be asserted that the operator in the left side of (2.12) acts from the total space of quadratically summable sequences l_2 into l_2 and is completely continuous there. Therefore, if the main determinant Δ of the system (2.12) is different from zero, then the Hilbert theorem about its solvability /5/ is applicable. Moreover, taking account of (2.10), we find from (1.5)

$$R_1 = P_0 + \sum_{i=1}^\infty P_i z_i(t), \quad P_0 = \int_{-1}^1 q_0(x)\,dx \tag{2.13}$$

$$P_i = \int_{-1}^1 q_i(x)\,dx = \pi\sqrt{2}\,\alpha_i\delta_i a_0^{(i)} = 0, \quad a_0^{(i)} = 0 \quad (i \geqslant 1)$$

The second of conditions (2.13) is to determine the unknown quantities α_i. Indeed, from the system (2.12) we have $a_0^{(i)} = \Delta_1\Delta^{-1}$, where Δ_1 is an auxiliary determinant obtained from Δ by replacing the first column by the elements $\{1, 0, \ldots, 0, \ldots\}$. The determinant Δ_1 is symmetric, consequently its roots $\alpha = \alpha_i\ (i \geqslant 1)$ are real. Have determined α_i we then find $a_j^{(i)}\ (j = 1, 2, \ldots)$ from the infinite algebraic system (2.12), and we therefore construct the sequence of functions $\{q_i(x)(\pi\sqrt{2}\alpha_i\delta_i)^{-1}\}$.

We now satisfy the selection of a countable set of constants δ_i and $z_i(1)\ (i \geqslant 1)$ for the integral equation (1.6) $(\alpha(1) \equiv 0)$. Assuming $g(x) \in L_2(-1, 1)$, we represent it in the form

$$g(x) = \sum_{n=0}^\infty g_n P_{2n}^*(x) \tag{2.14}$$

Substituting (2.14), and (2.11) into (1.6), we obtain

$$cX_j + \sum_{n=0}^\infty r_{jn} X_n = \pi[\sqrt{2}\,\delta(1)\,\delta_{0j} - g_j] \quad (j = 0, 1, \ldots) \tag{2.15}$$

$$X_j = \int_{-1}^1 q(x, 1) P_{2j}^*(x)\,dx \tag{2.16}$$

Having solved the infinite algebraic system (2.15), by taking account of the formula

$$q(x, 1) = q_0(x) + \pi\sqrt{2} \sum_{i=1}^\infty \alpha_i\delta_i z_i(1) \sum_{j=0}^\infty a_j^{(i)} P_{2j}^*(x)$$

we will have from the relationship (2.16)

$$\mathbf{Bz}(1) = \mathbf{b}, \quad \mathbf{b} \in l_2 \tag{2.17}$$

$$\mathbf{B} = \pi\sqrt{2}\,\|\alpha_i\delta_i a_j^{(i)}\|, \quad \mathbf{b} = \left\{X_j| - \int_{-1}^1 q_0(x) P_{2j}^*(x)\,dx\right\}$$

$$\mathbf{z}(1) = \{z_i(1)\}$$

$$(i = 1, 2, \ldots; j = 0, 1, 2, \ldots)$$

We now solve the system (2.17). Firstly, taking account of the results in /6/, it can be asserted that $\beta_i < \alpha_i < \beta_{i+1}$ $(i \geqslant 1)$, where β_i are characteristic primes of the operator A (2.7). But then $\alpha_i = O\,[i\,(\ln i)^{-1}]$ as $i \to \infty$. Secondly, setting $\delta_i = \alpha_i^{-1/2}$ (a foundation will be given below), by virtue of (2.12)

$$\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}[\alpha_i\delta_i a_j^{(i)}]^2 < \infty$$

i.e., the operator B is completely continuous from l_2 into l_2.

The element $\mathbf{z}^{(1)}(1) \in M$ (the set of uniformly bounded and equipotentially continuous sequences in l_2) will be called a quasisolution /7,8/ of (2.17) in M if

$$\inf\{\|\mathbf{B}\mathbf{z}^{(1)}(1) - \mathbf{b}\|_{l_2} : \mathbf{z}^{(1)}(1) \in M\}$$

Besides (2.17) we introduce the truncated system

$$\mathbf{B}^*\mathbf{z}^*(1) = \mathbf{b}^* \tag{2.18}$$

$$\mathbf{B}^* = \pi\sqrt{2}\,\|\alpha_i\delta_i a_j^{(i)}\|, \quad \mathbf{b}^* = \left\{X_j \Big| - \int_{-1}^{1} q_0(x)\,P_{2j}{}^*(x)\,dx\right\}$$

$$\mathbf{z}^*(1) = \{z_i(1)\}$$

$$(i = 1, 2, \ldots, n; \quad j = 0, 1, 2, \ldots, n-1)$$

It is proved /7,8/ that if the operator $\mathbf{B}^{-1}$ (not necessarily bounded) exists, then the quasisolution of (2.17) in the compact M also exists, is unique, and depends continuously on the right side $\mathbf{b}$. Moreover

$$\lim_{n\to\infty}\|\mathbf{z}^{(1)}(1) - \mathbf{z}^*(1)\|_{l_2} = 0$$

and $\mathbf{z}^*(1)$ can be found by the methods of /7,8/, say.

Furthermore, it follows from (2.10)

$$(q_i, q_j)_{L_2(-1,1)} = 2\pi^2(\alpha_i\alpha_j)^{-1/2}\sum_{n=0}^{\infty}a_n^{(i)}a_n^{(j)} \leqslant \tag{2.19}$$

$$2\pi^2(\alpha_i\alpha_j)^{-1/2}\left\{\sum_{n=0}^{\infty}[a_n^{(i)}]^2\sum_{n=0}^{\infty}[a_n^{(j)}]^2\right\}^{1/2} \leqslant 2\pi^2(\alpha_i\alpha_j)^{-1/2}B_2$$

$$B_2 = \text{const}; \quad i, j = 1, 2, \ldots$$

We note that $\delta(1)$ in the system (2.15) can be considered independent of δ_i $(i \geqslant 1)$ because

$$\delta(1) = \delta_0 + \sum_{i=1}^{\infty}\delta_i y_i(1)$$

and is determined during the solution of the problem in terms of the value of the impressing force R_1 by using the first statics condition (1.7) /3/. It is here pertinent to note that because of the first condition in (2.13) the constant δ is also related to R_1 (compare (2.9) to (1.6)).

3. We turn to an investigation of the Volterra integral equation of the second kind (2.5). According to the constraints imposed on its kernel in Sect.1, it is solvable uniquely /5/ in the space of functions $C(1, T)$ continuous in $[1, T]$ for any values of the parameters α_i and c. To construct the approximate solution of (2.5), we limit ourselves to the first two terms /1/ in the epxressions for $\varphi^{(k)}(\tau)$ and $f^{(k)}(t-\tau)$, i.e.,we take

$$f^{(k)}(t-\tau) = 1 - \exp[-\gamma^{(k)}(t-\tau)] \tag{3.1}$$

$$\varphi^{(k)}(\tau) = C_0{}^{(k)} + A^{(k)}\exp(-\beta^{(k)}\tau), \quad \beta^{(k)} = \text{const}$$

It is known /1/ that the solution (3.1) of the integral equation (2.5) can be obtained by reducing it to a certain ordinary second order differential equation with variable coefficients. Furthermore, assuming for simplicity that $\gamma^{(1)} = \gamma^{(2)} = \gamma$, $\beta^{(1)} = \beta^{(2)} = \beta$, $A^{(1)} = A^{(2)} = A^*$, $C_0{}^{(1)} = C_0{}^{(2)} = C_0$, we will have

$$z_i^{\cdot\cdot}(t) + \gamma\,[1 + C_0 + \mu_i(\tau_1{}^{(1)}, \tau_1{}^{(2)})\,e^{-\beta t}]\,z_i^{\cdot}(t) = 0 \tag{3.2}$$

$$z_i^{\cdot}(1)\,z_i^{-1}(1) + \gamma C_0 + \gamma e^{-\beta}\mu_i(\tau_1{}^{(1)}, \tau_1{}^{(2)}) = 0$$

$$\mu_i(\tau_1{}^{(1)}, \tau_1{}^{(2)}) = A^*(1 + \alpha_i c)^{-1}\,[\exp(\beta\tau_1{}^{(2)}) + \alpha_i\,c\exp(\beta\tau_1{}^{(1)})]$$

or

$$z_i(t) = z_i(1) - z_i(1)\,\gamma\,[C_0 + e^{-\beta}\mu_i(\tau_1^{(1)}, \tau_1^{(2)})] \times \tag{3.3}$$

$$\int_1^t \exp\left\{-\frac{\gamma}{\beta}[\beta(1 + C_0)(\tau - 1) + e^{-\beta}\mu_i(\tau_1^{(1)}, \tau_1^{(2)})(1 - e^{-\beta(\tau-1)})]\right\}d\tau$$

Having determined the function $z_i(t)$ according to (3.3), the sequence $\{y_i(t)\}$ can be constructed from (2.6), which we write in the following form by taking account of the relationships (1.1) and (1.2)

$$y_i(t) = z_i(t) + z_i(1)\,C^{(2)}(t-\tau_1^{(2)}, 1-\tau_1^{(2)}) + \int_1^t z_i^{\cdot}(\tau)\,C^{(2)}(t-\tau_1^{(2)}, \tau-\tau_1^{(2)})\,d\tau, \quad y_i(1) = z_i(1) \tag{3.4}$$

i.e., in addition the solution of the formulated problem $q(x, t)$ and $\delta(t)$ can also be written down. For the final foundation of the solution constructed, the convergence of the series (2.2) and (2.3) as well as the linear independence of the system of functions $\{y_i(t)\}$ should be proved.

Theorem 1. The system of functions $\{y_i(t)\}$ is linearly independent.

We shall reason for any finite system of functions $\{z_i(t)\}$ and we shall assume it linearly independent, i.e., such constants D_j exist among which are some different from zero, and the following equality is satisfied

$$\sum_{j=1} D_j z_j(t) \equiv 0 \tag{3.5}$$

Then taking account of (2.5) and the behavior of the creep kernels for a sufficiently large time, we write

$$\sum_{j=1} \frac{F(\tau_1^{(1)}, \tau_1^{(2)}) + c\alpha_j}{1 + c\alpha_j} D_j z_j(t) = 0 \tag{3.6}$$

From (3.5) and (3.6) we will have

$$\sum_{j=n} \frac{F(\tau_1^{(1)}, \tau_1^{(2)}) + c\alpha_j}{1 + c\alpha_j} D_j z_j(t) \equiv 0 \quad (n = 1, 2, \ldots)$$

This last equation is satisfied if and only if all $D_j \equiv 0$. The contradiction obtained indeed proves the linear independence of the system of functions $\{z_i(t)\}$.

The linear independence of the functions $\{y_i(t)\}$ follows from (3.3) and (3.4).

Theorem 2. The series (2.2) converges uniformly in t in $[1, T]$ for all $T > 1$ in $L_2(-1, 1)$ and determines the generalized solution of (1.4).

In fact, we estimate the residue of the series

$$\Big\| \sum_{n=j}^{\infty} z_n(t)\,q_n(x) \Big\|^2_{L_2(-1,1)} \leqslant \sum_{m,n=j}^{\infty} z_n(t)\,z_m(t)\,(q_m, q_n)_{L_2(-1,1)} \leqslant \tag{3.7}$$
$$2\pi^2 B_2 \sum_{n=j}^{\infty} |\alpha_n^{-1/2} z_n(t)| \sum_{m=j}^{\infty} |\alpha_m^{-1/2} z_m(t)| \leqslant \varepsilon \quad (j \to \infty)$$

where ε is an arbitrarily small number. The inequality (2.19), the fact of the boundedness of $z_i(t)\,[z_i(1)]^{-1}$, as well as the reasoning presented at the end of Sect.2 are used here.

If the inequality (3.7) is satisfied, then the series (2.2) converges uniformly at $t \in [1, T]$ $(T \geqslant 1)$ in $L_2(-1, 1)$ and the first property of the generalized solution is satisfied (see /9/, p.500). As follows from the reasoning in Sect.2, its remaining properties are also conserved. The theorem is proved. (The convergence of the series (2.3) is established analogously).

4. As an illustration we present the solution of the formulated problem in the case when $g(x) \equiv 0$ (the stamp has a flat base).

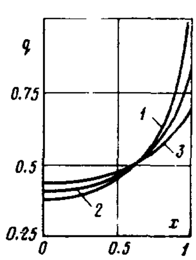

Fig.2

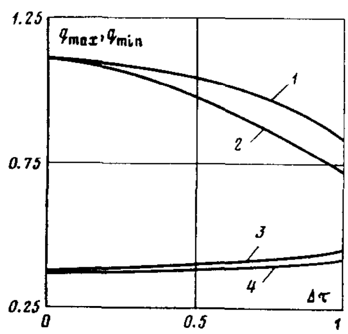

Fig.3

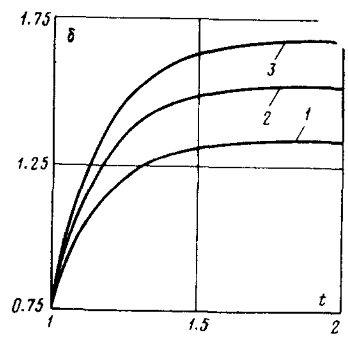

Fig.4

$$e = \alpha(t) \equiv 0;\ R_1 = 1;\ Ha^{-1} = 6,\ c = 0.5;\ C_0 = 0.5522;\ A^* = 4;\ \gamma = 6, \beta = 3.1;\ \tau_0 = 100 \text{ days}$$

Such values of the parameters are encountered for concrete structures analyses.

Graphs of the distribution of the contact pressures $q(x, t)$ are presented in Fig.2 for $\Delta\tau = \tau_1^{(1)} - \tau_1^{(2)} = 1$ and $t = 1$ (*1*), $t = 1.05$ (*2*), $t = 2$ (*3*). It is seen that as the time grows and the other factors remain unchanged the normal stresses under the stamp tend to a certain limit value.

The dependences between $q_{max}(t, \Delta\tau) = q(1, t, \Delta\tau)$, $q_{min}(t, \Delta t) = q(0, t, \Delta\tau)$ and $\Delta\tau$ are displayed in Fig.3 for different fixed values of t

$$(q_{max}(1.05, \Delta\tau) - (1), q_{max}(2, \Delta\tau) - (2), q_{min}(2, \Delta\tau) - (3), q_{min}(1.05, \Delta\tau) - (4)).$$

It can be noted that as $\Delta\tau$ grows the maximal contact pressures diminish while the minimal pressures increase.

The dependence $\delta(t)$ is presented in Fig.4 for fixed values of $\Delta\tau$ ($\Delta\tau = 0 - (1)$, $\Delta\tau = 0.8 - (2)$, $\Delta\tau = 1 - (3)$). It is seen that as the time t lapses the function $\delta(t)$ grows and tends to the limit value which will be the larger the greater the $\Delta\tau$.

Remarks. 1°. In the case when the layers are fabricated from the same material and have the very same age but the force acting on the stamp with a flat base is independent of time, by taking account of the results in /2/ and the correspondence principle /1/ we obtain that the contact pressure distribution will be the same as in the elastic problem, i.e., in this case the creep exerts no influence on the contact pressure distribution.

2°. The solution (3.1) of the integral equation (2.5) can be obtained in closed form even for arbitrary aging functions $\varphi^{(K)}(\tau)$. In particular, for $\gamma^{(1)} = \gamma^{(2)} = \gamma$ we will have

$$z_i(t) = z_i(1)\left[1 + \int_1^t R_i(t, \tau)\, d\tau\right]$$

where $R_i(t, \tau)$ is the resolvent of the N. Kh. Arutiunian kernel

$$K_i(t, \tau) = \frac{1}{1 + c\alpha_i} \frac{\partial}{\partial\tau} \{(1 - e^{-\gamma(t-\tau)}) [\varphi^{(2)}(\tau - \tau_1^{(2)}) + \alpha_i c \varphi^{(1)}(\tau - \tau_1^{(1)})]\}$$

whose form is presented in /10/.

The authors are grateful to N. Kh. Arutiunian and V.M. Aleksandrov for attention to the research.

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Translated by M.D.F.